# *Schrödinger's Operator and the zeros of the eta function*

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*Summary*

*Physicists study quantum systems, like free particles, using* **Schrödinger** *operator. Here, I will expose some ideas turning around the known relation between the roots of the zeta function and random matrices and their eigenvalues.*


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The **Schrödinger** equation is a linear operator, one of its parts acts as a ***Hamiltonian*** $\widehat{H}$ on a wavefunction $\Psi$, and is presented as a **Hermetian** matrix

$$\overline{H^T} = H$$

the wave function $\Psi$, gives essential information, that is measurements about a particle, these measurements ( position, momentum, energy…) are known to be the eigenvalues of $H$ . The squared amplitude of the wave function for $X = (\mathrm{x}, \mathrm{y}, \mathrm{z})$ and time $t$:

$$\|\Psi(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t})\|^2$$

is the probability that a particle has a measurement $M$ at a time $t$.

Ordinary *Wave equation* in two space dimensions has the form

$$\Delta u = \frac{1}{\alpha^2} u_{tt}$$

it has some relation with two other known equations namely :

$$\Delta u = 0$$

$$\Delta u = |k| u_t$$

Since there is an interesting relation between the roots of zeta function $\sigma_n + i\gamma_n$ and the eigenvalues of random matrices, a relation that is much studied by mathematicians and physicists since **Montgomery**'s discovery, I try here , through it, to speculate on some possible relation between: the Riemann hypothesis $\boldsymbol{RH}$ , the potential $V$ , and the zeta function as a part of the wave

function $\psi$ .

## The function $\boldsymbol{F(t)}$

The *eta function* is defined for $t \in \mathbb{R}$ and $\sigma > 0$ as the infinite series:

$$\eta(z) = \sum_{1}^{+\infty} \frac{(-1)^{n-1}}{n^z}$$

or in two infinite series

$$\eta(\sigma + it) = \sum\nolimits_{n\geq 1} \frac{\cos(tlogn)}{n^\sigma}(-1)^{n-1} - i\sum\nolimits_{n\geq 1} \frac{\sin(tlogn)}{n^\sigma}(-1)^{n-1}$$

any complex function is also written as

$$\eta(\sigma + it) = \Re\eta(\sigma, t) + i\Im\eta(\sigma, t)$$

The eta function has the same *non trivial* roots of the zeta function $\zeta$, and if we assume the veracity of the $\boldsymbol{RH}$, they are all of the form:

$$\frac{1}{2} + i\gamma_n$$

## Waves as a string's vibrations

As the electron is seen as a wave according to the **de Broglie** equation(1924) $\lambda = \frac{h}{p}$ , it is known that the **Schrödinger** equation (1927) , or operator:

$$i\hbar\psi_t = \widehat{H}(\psi)$$

$$\psi(X = (x, y, z), \mathrm{t}) = f + ig$$

came to model the *wave function* $\psi$ .

As we know from the theory of analytic functions , both $\Re\eta$ and $\Im\eta$ are harmonic functions, verifying the **Laplace** PDE :

$$\frac{\partial^2 \Re\eta(\sigma, t)}{\partial \sigma^2} + \frac{\partial^2 \Re\eta(\sigma, t)}{\partial t^2} = 0$$

Where :

$$\frac{\partial^2 \Re\eta(\sigma, t)}{\partial \sigma^2} = \sum\nolimits_{n\geq 1} \frac{\cos(tlogn)}{n^\sigma} \times log^2 n \times (-1)^{n-1}$$

$$\frac{\partial^2 \Re\eta(\sigma, t)}{\partial t^2} = -\sum\nolimits_{n\geq 1} \frac{\cos(tlogn)}{n^\sigma} \times log^2 n \times (-1)^{n-1}$$


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Nothing prevents us to see $\Delta u = 0$ as a *complex wave equation*:

$$(\Re\eta)_{\sigma\sigma} = i^2(\Re\eta)_{tt}$$

otherwise as

$$i(\Re\eta)_{\sigma\sigma} = -i(\Re\eta)_{tt}$$

For simplicity, we consider here the **Schrödinger** equation for a plane wave (only $X = x$), where $\widehat{H}$ is the **Hamiltonian**

$$\widehat{H} = -\frac{\hbar^2}{2m}\Delta + \mathrm{V}$$ [1]

We use one dimension wave function $\psi(\mathrm{x},\mathrm{t})$, and for convenience, we have to interchange the variables in our infinite series $\eta(z)$:

$t$ of $\Re\eta$ is the space $x$ of $\psi : -\infty < t < +\infty$

$\sigma$ of $\Re\eta$ is the time $t$ of $\psi : 0 < \sigma < +\infty$

the equations gives

[1] " $(r,t)$ *designates a potential energy here. For example, it may be the product of an electric potential and the particle's charge. In quantum mechanics,* $\boldsymbol{V}(r,t)$ *is commonly called a potential.* " $\boldsymbol{Tannoudji - Diu - Laloe}$ (1/20).

$$\begin{cases} \hbar\psi_\sigma = (\Re\eta)_{\sigma\sigma} & \rightarrow \quad \hbar\psi = (\Re\eta)_\sigma + i\boldsymbol{F}(\boldsymbol{t}) \\ \hat{H}(\psi) = -i(\Re\eta)_{tt} & \rightarrow \hat{H}\left(\frac{(\Re\eta)_\sigma + iF(t)}{\hbar}\right) = -i(\Re\eta)_{tt} \end{cases}$$

Then the function to be found is

$$\boldsymbol{F}(\boldsymbol{t})$$

and we get an expression for the wave function:

$$\boxed{\psi(t,\sigma) = \frac{(\Re\eta)_\sigma(\sigma,t) + i\boldsymbol{F}(t)}{\hbar}}$$

or more exactly:

$$\psi(\sigma,t) = \frac{i\boldsymbol{F}(t)}{\hbar} + \frac{1}{\hbar}\sum\nolimits_{n\geq 1} \frac{cos(tlogn)}{n^\sigma} \times logn \times (-1)^{n-1}$$

with , of course, the *normalization* condition:

$$\int_{\mathbb{R}} \|\psi\|^2 dt = 1$$

otherwise :

$$\int_{\mathbb{R}} \left| \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^{\sigma}} \times logn \times cos(tlogn) + i\boldsymbol{F}(t) \right|^2 dt = \hbar^2$$

If $\int_0^T (\Re\eta)_\sigma{}^2 dt$ converges at all at $+\infty$. For $(\Re\eta)_\sigma$ we know that:

$$\int_0^T (\Re\eta)_\sigma dt = \sum_1^{+\infty} \frac{(-1)^{n-1}}{n^{\sigma}} \times \sin(Tlogn) < +\infty$$

If there is a problem of convergence, then to avoid it:

i) We can consider integration in $[0, T]$ for $T > 0$ .

ii) The probability of the existence of a particle diminishes adequately to 0 as $T \to +\infty$ .

Anyway , we get

$$\int_{-T}^{T} \boldsymbol{F}(t)^2 dt = \hbar^2 - \int_{-T}^{T} (\Re\eta)^2{}_\sigma dt$$

Since $cos$ is even then :

$$\int_0^T \boldsymbol{F}(t)^2 dt = \frac{\hbar^2}{2} - \int_0^T (\Re\eta)_\sigma{}^2 dt$$

Or otherwise



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$$\int_0^T \boldsymbol{F}(t)^2 dt = \frac{\hbar^2}{2} - \int_0^T (\Re[\,(1-2^{1-z})\zeta(z)])_\sigma{}^2 dt$$

There is some inconsistency here , namely that $\hbar^2$ is *very small*. But we can avoid this inconsistency by thinking differently : we search for possible zeros, as $T$ grows, of

$$\frac{\hbar^2}{2} - \int_0^T (\Re[\,(1-2^{1-z})\zeta(z)])_\sigma{}^2 dt$$

which , of course, should not be simple zeros (since $\int_0^T \boldsymbol{F}(t)^2 dt \geq 0$ ), and ask for their meaning in this situation.

## The Hamiltonian in action

Applying the operator on $\psi$

$$(-\frac{\hbar^2}{2m}\Delta + \mathbf{V})\left(\frac{(\Re\eta)_\sigma + i\boldsymbol{F}(t)}{\hbar}\right) = -i(\Re\eta)_{tt}$$

gives the equation



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$$-\frac{\hbar^2}{2\hbar m}\left((\Re\eta)_{\sigma tt}+iF_{tt}(t)\right)+\frac{1}{\hbar}\boldsymbol{V}((\Re\eta)_\sigma+i\boldsymbol{F}(t))=-i(\Re\eta)_{tt}$$

where

$$(\Re\eta)_{\sigma tt}=-\sum\nolimits_{n\geq 1}\frac{cos(tlogn)}{n^\sigma}\times log^3n\times(-1)^{n-1}$$

if we want the equation to have sense, then it must verify ***two conditions*** :

$$\begin{cases}\dfrac{\hbar^2}{2m}\boldsymbol{V}((\Re\eta)_\sigma)+\boldsymbol{V}\left(\boldsymbol{F}(t)\right)=-\hbar(\Re\eta)_{tt}\\ \\ -\dfrac{\hbar^2}{2m}(\Re\eta)_{\sigma tt}+\boldsymbol{V}((\Re\eta)_\sigma)=0\end{cases}$$

The ***first condition***

We know that

$$\eta'(z)=(\Re\eta)_\sigma+i(\Im\eta)_\sigma=(\Im\eta)_t-i(\Re\eta)_t$$

and

$$\zeta'(z)=\frac{\eta'(z)}{(1-2^{1-z})}+\eta(z)\times\left(\frac{1}{1-2^{1-z}}\right)'$$


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So in the critical strip (let $\sigma = 1$ $alone$) $\eta'(z)$ and $\zeta'(z)$ have the same zeros. And if $\boldsymbol{RH}$ is true then $\eta'(z)$ has no roots in $0 < \sigma < \frac{1}{2}$ because it has been proved that $\boldsymbol{RH}$ is equivalent to the fact that $\zeta'(z)$ has no roots when $0 < \sigma < \frac{1}{2}$.

So in the first condition, the term$(\Re\eta)_\sigma$ vanishes at the zeros of $\eta'(z)$ and $\zeta'(z)$. The term $(\Re\eta)_{tt}$ vanishes too for $\sigma = \sigma_0$ and (using **Schwarz**) its zeros are between different vertical sets of zeros. What relation has this first condition with $\boldsymbol{RH}$ ?

So when does:

$$\boldsymbol{V}\big(\boldsymbol{F}(t)\big) = 0 \qquad ?$$

Here $V$ is independent of time. What is the meaning of this in physics?

The *__second condition__*

We have seen that the term$(\Re\eta)_\sigma$ vanishes at the zeros of $\eta'(z)$ and $\zeta'(z)$. The term $(\Re\eta)_{\sigma tt}$ vanishes at the zeros of $\eta'(z)$ and $\zeta'(z)$, and this for $\sigma = \sigma_0$ and its zeros are also between different vertical sets of zeros. So the equation

$$\boldsymbol{V}((\Re\eta)_\sigma) = \frac{\hbar^2}{2m}(\Re\eta)_{\sigma tt}$$

Cannot be identically zero.